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Jordan Algebras and F_4 Bundles over the Affine Plane

R. Parimala,* V. Suresh,† and Maneesh L. Thakur‡

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay, 400 005, India

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INTRODUCTION

Let k be a field of characteristic different from 2 and 3. Let J be a finite dimensional exceptional central simple Jordan algebra over k. Let G be the group of automorphisms of J. Then G is an algebraic group of type F_4 defined over k. Let X be an integral scheme over k. The set $H^1(X, G)$ classifies the isomorphism classes of principal G-bundles on X. This set also classifies the isomorphism classes of exceptional simple Jordan algebra bundles on X.

We assume that G is anisotropic and that the Jordan algebra J is reduced. In this paper, we construct an infinite family $J^{[i]}$ of Jordan algebra bundles over the affine plane $\mathbb{A}_k^2$ which are mutually non-isomorphic and which specialise to J at any rational point. These bundles have the rigidity property that the associated principal G-bundles $P_{J^{[i]}}$ admit no reduction of the structure group to any proper connected reductive subgroup of G. A theorem of the same kind for principal G-bundles, G connected reductive anisotropic, with the exceptions of G_2 and F_4, was proved by Raghunathan in [R]. The case $G = \operatorname{Aut} J$, with J a division Jordan algebra, will be dealt with elsewhere, thus settling the case of F_4.

Let k be as above and C a Cayley algebra over k. Let $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ be a diagonal matrix with $\gamma_i \in k^*$. Let $\sigma : M_3(C) \to M_3(C)$ be the involution given by $\sigma(X) = \Gamma^{-1} \overline{X}^t \Gamma$, bar denoting the entrywise action of the canonical involution on C. Let $J = \mathsf{H}_3(C, \Gamma)$ be the set of symmetric

* E-mail address: parimala@tifrvax.tifr.res.in.
† E-mail address: suresh@tifrvax.tifr.res.in.
‡ E-mail address: maneesh@tifrvax.tifr.res.in.

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elements for σ. The multiplication $A.B = \frac{1}{2}(AB + BA)$ makes J into a Jordan algebra over k. Every reduced exceptional Jordan algebra is constructed this way. As a sequel to [KPS2], with the assumption that C is a Cayley division algebra, an infinite family of Cayley algebra bundles C_i on $\mathbb{A}_k^2$ was constructed in [T] with the property that the quadratic bundle C_i' of trace zero elements in C_i is indecomposable. Then $\{J_i = H_3(C_i, \Gamma)\}$ is a family of Jordan algebra bundles on $\mathbb{A}_k^2$ which specialize to J at a rational point. Suppose that the group $G = \operatorname{Aut} J$ is anisotropic (cf. (3.1)). Then C is a division algebra. Using the algebras J_i as prototypes on open subsets of $\mathbb{A}_k^2$ and using patching techniques of [P], we construct an infinite family of Jordan algebra bundles $J^{[i]}$ over $\mathbb{A}_k^2$ with the following properties

(i) $J^{[i]}$ specialises to J at any point.

(ii) The associated principal G-bundle $P_{J^{[i]}}$ admits no reduction of the structure group to any proper connected reductive subgroup of G.

Certain rigidity properties for H-bundles, where H is a connected reductive group, and in particular for the orthogonal bundles, are proved in Section 2, which are used in subsequent sections for proving the rigidity of the G-bundles, with G as above.

We remark that, since for k a number field or the field of real numbers, every exceptional simple Jordan algebra over k is reduced, the proof for the case F_4 of Raghunathan's theorem is complete for these fields.

1. RIGID FORMS

Let (E, q) be a quadratic space over a scheme X. We call q *rigid* if the orthogonal group of q consists only of $\pm$ identity. We owe this terminology to W. Scharlau, who called such forms "starr." In this section we give a characterisation of rigid spaces over the projective plane $\mathbb{P}_k^2$.

Let N be an indecomposable vector bundle over $\mathbb{P}_k^2$. Then $N = \operatorname{End} N$ is a finite dimensional local k-algebra and $D = N/\operatorname{rad} N$ is a division algebra over k. In fact N is absolutely indecomposable (i.e., $N_{\bar{k}}$ is indecomposable, $\bar{k}$ denoting the algebraic closure of k) if and only if $D = k$. Suppose N is self dual and $\alpha : N \to N^*$ is an isomorphism. Since N is local, replacing α by $\alpha \pm \alpha^t$, we may assume that $\alpha^t = \epsilon\alpha$ with $\epsilon = \pm 1$. The isomorphism α gives rise to an involution $\tau : N \to N$ given by $\tau(f) = \alpha f^t \alpha^{-1}$ which induces an involution $\tau : D \to D$. We recall [QSS, Theorem 3.3] the equivalence between the categories of quadratic spaces over $\mathbb{P}_k^2$ with underlying vector bundle isotypical of type N and ϵ-hermitian spaces over (N, τ). Let $E = \bigoplus_{l\ copies} N$ be a vector bundle on $\mathbb{P}_k^2$ with a symmetric isomorphism $q : E \to E^*$. The free N-module $E = \operatorname{Hom}(N, E)$ supports the ϵ-hermitian form $\tilde{q}: E \to \mathrm{Hom}_N(E, N)$ defined by

$$\tilde{q}(f)(g) = \alpha^{-1} \circ g^t \circ q \circ f.$$

The assignment $(E, q) \mapsto (E, \tilde{q})$ yields the required equivalence. We also recall [QSS, Theorem 3.3] that every ϵ-hermitian space over (N, τ) is determined up to isometry by its reduction modulo rad N, which is a space over (D, τ).

Suppose $\alpha : N \to N^*$ is symmetric. Using the above equivalence, combined with reduction modulo rad N, one gets the following

LEMMA 1.1. *The set of isometry classes of quadratic spaces, with N as the underlying vector bundle, is in bijection with the set $\{u \in D^* | \tau u = u\}/\sim$, where $\sim$ is the unitary equivalence $u \sim u' \Leftrightarrow$ there exists $x \in D^*$ such that $u' = xu\tau(x)$. In particular, if N is absolutely indecomposable, every quadratic space on N is isometric to $\lambda\alpha$ for some $\lambda \in k^*$.*

Let (E, q) be a quadratic space over $\mathbb{P}_k^2$. Let $E = \oplus F_i$ be the decomposition of E into isotypical components F_i of type N_i. If $q : E \to E^*$ pairs F_i with F_j for $i \neq j$ then $N_i \simeq N_j^*$ and hence $N_i \not\simeq N_i^*$. In this case q splits off hyperbolic summands. Suppose (E, q) is anisotropic (i.e., generically anisotropic). Then (E, q) decomposes as an orthogonal sum $\perp (F_i, q_i)$ where $q_i = q|_{F_i}$.

Let (F, q) be an anisotropic space with F isotypical of type N, i.e., $F = \bigoplus_{l\ copies} N$. Since N is self dual, one may choose an isomorphism $\alpha : N \to N^*$ with $\alpha^t = \pm\alpha$. If $N/\mathrm{rad}\, N \neq k$ (i.e., N is not absolutely indecomposable), one can always choose $\alpha : N \to N^*$ such that $\alpha^t = \alpha$. Suppose $N/\mathrm{rad}\, N = k$. If $\alpha^t = -\alpha$, isometry classes of quadratic spaces on F correspond bijectively with isometry classes of skew-symmetric spaces of rank l over k. Since every non-degenerate skew-symmetric form over k is hyperbolic, $l = 2m$ and there is a unique quadratic space structure on F, namely the hyperbolic space $\bigoplus_m H(N)$. This contradicts the fact that (F,q) is anisotropic. Thus $\alpha^t = \alpha$ in this case. Therefore isometry classes of quadratic spaces over F correspond bijectively with isometry classes of hermitian spaces over (D, τ). Since every hermitian form over (D, τ) is diagonalisable, it follows, using the equivalence of [QSS, Theorem 3.3], that $(F, q) \simeq \perp (N, q_i)$ for quadratic spaces q_i on F. Thus, we arrive at the following

PROPOSITION 1.2. *Let (E, q) be an anisotropic indecomposable quadratic space over $\mathbb{P}_k^2$. Then E is indecomposable as a vector bundle.*

PROPOSITION 1.3. *Let (E, q) be an anisotropic quadratic space over $\mathbb{P}_k^2$. Then $O(q) = \{\pm 1\}$ (i.e., q is rigid) if and only if E is absolutely indecomposable.*

Proof. Suppose $O(q) = \{\pm 1\}$. Then clearly q is indecomposable as a quadratic space. By (1.2), $\mathcal{E}$ is indecomposable as a vector bundle. Let $E = \operatorname{End} \mathcal{E}$ and $D = E/\operatorname{rad} E$ and τ the involution on E and D, induced by q. If $\mathcal{E}$ is not absolutely indecomposable, then $D \neq k$. There exists $u \in D^*$, $u \neq \pm 1$, such that $u\tau(u) = 1$. Since $\operatorname{rad} E$ is nilpotent, u lifts to a unit $\tilde{u} \in \operatorname{End} \mathcal{E}$ such that $\tilde{u}\tau(\tilde{u}) = 1$, i.e., $\tilde{u}q\tilde{u}^t q^{-1} = 1$, i.e., $\tilde{u} \in O(q)$ and $\tilde{u} \neq \pm 1$, contradicting the assumption on q. Thus $\mathcal{E}$ is absolutely indecomposable.

Suppose conversely that $\mathcal{E}$ is absolutely indecomposable. Let $u \in O(q)$. Then $u\tau(u) = 1$. Since $E/\operatorname{rad} E = k$ and τ is identity on k, $(\bar{u})^2 = 1$, bar denoting reduction modulo $\operatorname{rad} E$. Replacing u by $-u$, if necessary, we may assume that $\bar{u} = 1$, i.e., $u = 1 + \eta$, $\eta \in \operatorname{rad} E$, i.e., $(u-1)^n = 0$ for some n. Thus u, as an element of the orthogonal group of $q \otimes k(X, Y)$, is unipotent. Since q is generally anisotropic, $O(q \otimes k(X, Y))$ has no nontrivial unipotent elements [Bo, 6.4.2]. Thus $u = 1$. This completes the proof. ∎

COROLLARY 1.4. *Let $(\mathcal{E}, q)$ be an indecomposable anisotropic quadratic space over $\mathbb{P}^2_k$ of rank p, p a prime. Then q is rigid.*

Proof. By (1.2), $\mathcal{E}$ is indecomposable as a vector bundle. Suppose that $\mathcal{E}_{\bar{k}}$ is decomposable. One concludes by looking at the Galois action on the decomposition of $\mathcal{E}_{\bar{k}}$ into direct sum of isotypical components, that $\mathcal{E}_{\bar{k}}$ is a direct sum of line bundles. Then clearly $\mathcal{E}$ itself is a direct sum of line bundles, contradicting its indecomposability. Thus $\mathcal{E}_{\bar{k}}$ is indecomposable and by (1.3), $O(q) = \{\pm 1\}$. ∎

2. RIGID FORMS OVER $\mathbb{A}^2_k$

We recall from [KPS, Theorem 2.1] that every anisotropic quadratic space over $\mathbb{A}^2_k$ extends uniquely to a quadratic space on $\mathbb{P}^2_k$. We begin with a corresponding statement on extension of isometries.

LEMMA 2.1. *Let R be a domain and p a prime in R. Let q and q' be quadratic spaces over R. Suppose that $q' \otimes R/(p)$ is anisotropic. Let $\alpha : q \otimes R[1/p] \to q' \otimes R[1/p]$ be an isometry. Then α extends uniquely to an isometry $\alpha : q \simeq q'$.*

Proof. Compare [KPS1, Proposition 1.1]. ∎

COROLLARY 2.2. *Let $\tilde{q}$ be an anisotropic quadratic space over $\mathbb{P}^n_k$ and q its restriction to $\mathbb{A}^n_k$. Then $O(q) = O(\tilde{q})$.*

Proof. Clearly, restriction yields an injection $O(\tilde{q}) \hookrightarrow O(q)$. Let $\alpha \in O(q)$. Let X_i, $0 \leq i \leq n$, be the homogeneous coordinates of $\mathbb{P}^n_k$ such that $\mathbb{A}^n_k = D(X_n)$. Let α_i denote the restriction of α to $D(X_i) \cap D(X_n)$. By

(2.1), α_i extends uniquely to an isometry $\tilde{\alpha}_i$ on $D(X_i)$. Let $\tilde{\alpha}_{ij}$ denote the restriction of $\tilde{\alpha}_i$ to $D(X_i) \cap D(X_j)$. Then $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}$ on $D(X_i) \cap D(X_j) \cap D(X_n)$ by (2.1) and hence they coincide. Thus the $\{\tilde{\alpha}_i\}$ patch on $D(X_i) \cap D(X_j)$ to yield an isometry $\tilde{\alpha} : \tilde{q} \to \tilde{q}$ which uniquely extends α. This proves the corollary. ∎

PROPOSITION 2.3. *Let q be an anisotropic quadratic space over $\mathbb{A}_k^2$ and $(\mathsf{E}, \tilde{q})$ its extension to $\mathbb{P}_k^2$. Then q is indecomposable if and only if E is indecomposable as a vector bundle. Further, q is rigid if and only if E is absolutely indecomposable.*

Proof. Since the extension $(\mathsf{E}, \tilde{q})$ of q is unique, if q decomposes then so does $\tilde{q}$ and hence E decomposes as a vector bundle. Conversely, if E decomposes as a vector bundle, by (1.2), $\tilde{q}$ being anisotropic, $\tilde{q}$ decomposes and hence q decomposes. Since $O(q) = O(\tilde{q})$, (2.2), the second assertion in the proposition is immediate from (1.3). ∎

COROLLARY 2.4. *Let q be an indecomposable quadratic space over $\mathbb{A}_k^2$ of prime rank. Then q is rigid (i.e., $O(q) = \{\pm 1\}$).*

Proof. This is immediate from (1.4) and (2.3). ∎

COROLLARY 2.5. *Let q_i be rigid quadratic spaces over $\mathbb{A}_k^2$ such that $\perp q_i$ is anisotropic. Suppose q_i and q_j are not similar for $i \neq j$. Then every isometry $\alpha : \perp q_i \to \perp q_i$ maps q_i to q_i.*

Proof. Since each q_i is rigid, if $(\mathsf{E}_i, \tilde{q}_i)$ denotes its extension to $\mathbb{P}_k^2$, E_i is absolutely indecomposable. In particular, each E_i supports a unique quadratic space structure up to scalars. Since q_i is not similar to q_j, $\tilde{q}_i$ is not similar to $\tilde{q}_j$. Hence $\mathsf{E}_i \not\simeq \mathsf{E}_j$ for $i \neq j$.

By (2.2), α extends to an isometry $\tilde{\alpha} : \perp (\mathsf{E}_i, \tilde{q}_i) \to \perp (\mathsf{E}_i, \tilde{q}_i)$. By the Krull–Schmidt theorem, $\tilde{\alpha}(\mathsf{E}_i) = \mathsf{E}_i$ so that $\alpha(q_i) = q_i$. ∎

We conclude this section by giving a construction of rigid spaces over $\mathbb{A}_k^2 = \operatorname{Spec}(k[X, Y])$. The idea of this construction goes back to the construction of rigid spaces over $\mathbb{A}_k^2$ given in [P]. Let $p_1, p_2 \in k[X]$ be polynomials such that $(p_1, p_2) = 1$. Let $(q_1, q_2), (q_3, q_4)$ be pairs of rigid quadratic spaces over $\mathbb{A}_k^2$ such that q_1, q_2 are not similar and q_3, q_4 are not similar. Let q_0, q_0' be quadratic forms over k such that $q_0 \perp q_0'$ is anisotropic and $\bar{q}_1 = \bar{q}_3 = q_0 \otimes k[X]$, $\bar{q}_2 = \bar{q}_4 = q_0' \otimes k[X]$, bar denoting the reduction modulo Y. Suppose that $\pi_1 : (q_1)_{p_1} \to (q_0)_{p_1}$, $\pi_2 : (q_2)_{p_1} \to (q_0')_{p_1}$ are isometries over $k[X]_{p_1}[Y]$ such that $\overline{\pi}_i =$ identity, $i = 1, 2$. Suppose $\pi_3 : (q_2)_{p_2} \to (q_0)_{p_2}$, $\pi_4 : (q_4)_{p_2} \to (q_0')_{p_2}$ are isometries over $k[X]_{p_2}[Y]$ such that $\overline{\pi}_i =$ identity, $i = 3, 4$. Let $\psi \in O(q_0 \perp q_0')$ be such that $\psi(q_0)$ and $\psi(q_0')$ are contained neither in q_0 not in q_0'. Let q be the quadratic space over $\mathbb{A}_k^2$ obtained by taking the space $(q_1 \perp q_2)_{p_2}$ over

$K[X]_{p_2}[Y]$ and $(q_3 \perp q_4)_{p_1}$ over $k[X]_{p_1}[Y]$ and patching them over the intersection $k[X]_{p_1p_2}[Y]$ by the isometry $\theta = (\pi_3 \perp \pi_4)^{-1} o\psi o(\pi_1 \perp \pi_2)$.

PROPOSITION 2.6. *The space q, constructed above, is rigid.*

Proof. Any isometry $\alpha : q \to q$ is given by a pair of isometries $\alpha_1 : (q_1 \perp q_2)_{p_2} \to (q_1 \perp q_2)_{p_2}$ over $k[X]_{p_2}[Y]$ and $\alpha_2 : (q_3 \perp q_4)_{p_1} \to (q_3 \perp q_4)_{p_1}$ over $k[X]_{p_1}[Y]$ such that the diagram

$$\begin{array}{ccc} (q_1 \perp q_2)_{p_1p_2} & \xrightarrow{\theta} & (q_3 \perp q_4)_{p_1p_2} \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ (q_1 \perp q_2)_{p_1p_2} & \xrightarrow{\theta} & (q_3 \perp q_4)_{p_1p_2} \end{array}$$

commutes. Let bar denote the reduction modulo Y. Then $\tilde{\theta} = \psi$ is defined over k and $\bar{\alpha}_2\psi = \psi\bar{\alpha}_1$ is defined over both $k[X]_{p_1}$ and $k[X]_{p_2}$. Since $(p_1, p_2) = 1$, $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are defined over $k[X]$. Since $q_1 \perp q_2$ is extended after inverting p_1 and α_1 is an isometry of $(q_1 \perp q_2)_{p_2}$ onto itself with $\bar{\alpha}_1$ defined over $k[X]$, by [P, 2.4], α_1 is defined over $k[X, Y]$. Since q_1 and q_2 are rigid and nonsimilar, by (2.5), $\alpha_1(q_1) = q_1$ and $\alpha_1(\mathrm{q}_2) = q_2$. Since the q_i are rigid, $\alpha_1 = \epsilon_1 \perp \epsilon_2$ where $\epsilon_i : q_i \to q_i$ is given by ± 1. Similarly $\alpha_2 = \epsilon_3 \perp \epsilon_4$ with $\epsilon_i = \pm 1$. Thus $\alpha_i = \bar{\alpha}_i$ and we have $(\epsilon_3 \perp \epsilon_4)\psi = \psi(\epsilon_1 \perp \epsilon_2)$. Writing

$$\psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{pmatrix}$$

as a transformation of $q_0 \perp q_0'$, we have, by the assumptions on ψ, $\psi_i \neq 0$ for $1 \leq i \leq 4$. Further,

$$\begin{pmatrix} \epsilon_3\psi_1 & \epsilon_3\psi_2 \\ \epsilon_4\psi_3 & \epsilon_4\psi_4 \end{pmatrix} = \begin{pmatrix} \epsilon_1\psi_1 & \epsilon_2\psi_2 \\ \epsilon_1\psi_3 & \epsilon_2\psi_4 \end{pmatrix}$$

from which it is immediate that $\alpha = \pm 1$. Therefore $O(q) = \{\pm 1\}$, so that q is rigid. ∎

3. SOME GENERALITIES ON REDUCED EXCEPTIONAL JORDAN ALGEBRAS

Let R be a commutative ring in which 2 is invertible. Let Zorn(R) denote the R-module $\left(\begin{smallmatrix} R & R^3 \\ R^3 & R \end{smallmatrix}\right)$. The multiplication

$$\begin{pmatrix} a & x \\ x' & a' \end{pmatrix}\begin{pmatrix} b & y \\ y' & b' \end{pmatrix} = \begin{pmatrix} ab - x.y' & ay + b'x + x' \times y' \\ bx' + a'y' + x \times y & a'b' - x'.y \end{pmatrix},$$

where x, y and $x \times y$ respectively denote the usual scalar and vector product on R^3, makes $\operatorname{Zorn}(R)$ into a non-associative R-algebra. This algebra has a canonical involution defined on it, namely,

$$A = \begin{pmatrix} a & x \\ x' & a' \end{pmatrix} \mapsto \overline{A} = \begin{pmatrix} a' & -x \\ -x' & a \end{pmatrix}.$$

One has $\overline{AB} = \overline{B}\,\overline{A}$ and $n(A) := A\overline{A} = \overline{A}A \in R$, $t(A) := A + \overline{A} \in R$. The map n is a non-singular quadratic form which is multiplicative, i.e.,

$$n(AB) = n(A)n(B), \qquad A, B \in \operatorname{Zorn}(R).$$

DEFINITION. A *Cayley algebra* over R is a non-associative R-algebra C such that for some faithfully flat extension S/R, $C \otimes_R S \simeq \operatorname{Zorn}(S)$ as S-algebras. The algebra C acquires a non-degenerative quadratic form n_C which we call the *norm* on C. It also has an involution bar with the property that

$$n_C(X) = X\overline{X} = \overline{X}X \in R, \qquad \text{for all } X \in C.$$

If R is a field, n_C is a 3-fold Pfister form and the algebra C is division if and only if n_C is anisotropic.

Let C be a Cayley algebra over R. For $\gamma_1, \gamma_2, \gamma_3$, units in R, let Γ denote the diagonal matrix $\langle \gamma_1, \gamma_2, \gamma_3 \rangle$. Let $H_3(C, \Gamma) = \{A \in M_3(C) \mid \Gamma^{-1}\overline{A}^t\Gamma = A\}$, where $M_3(C)$ denotes the set of 3×3 matrices with entries in C and bar denotes entrywise action of the involution bar on C. Explicitly we have

$$H_3(C, \Gamma) = \left\{ \begin{pmatrix} \alpha_1 & c & \gamma_1^{-1}\gamma_3\overline{b} \\ \gamma_2^{-1}\gamma_1\overline{c} & \alpha_2 & a \\ b & \gamma_3^{-1}\gamma_2\overline{a} & \alpha_3 \end{pmatrix} \middle| \begin{array}{l} a, b, c \in C, \\ \alpha_i \in R \end{array} \right\}.$$

We note that $H_3(C, \Gamma)$ is a projective R-module of rank 27.

Let AB denote the usual matrix product in $M_3(C)$. On $H_3(C, \Gamma)$ we define a multiplication by $A.B = \frac{1}{2}(AB + BA)$. This makes $H_3(C, \Gamma)$ into a non-associative commutative algebra over R. If R is a field, $H_3(C, \Gamma)$ is a reduced exceptional simple Jordan algebra and every reduced simple exceptional Jordan algebra is isomorphic to $H_3(C, \Gamma)$ for suitable C and Γ.

For

$$A = \begin{pmatrix} \alpha_1 & c & \gamma_1^{-1}\gamma_3\overline{b} \\ \gamma_2^{-1}\gamma_1\overline{c} & \alpha_2 & a \\ b & \gamma_3^{-1}\gamma_2\overline{a} & \alpha_3 \end{pmatrix} \in H_3(C, \Gamma),$$

one defines a trace function by $T(A) = \alpha_1 + \alpha_2 + \alpha_3$, the sum of the diagonal entries of A. Then $Q(A) = T(A^2)$ is a non-singular quadratic form on $H_3(C, \Gamma)$, called the *trace form*. In fact for A as above,

$$Q(A) = T(A^2) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\big(\gamma_3^{-1}\gamma_2 n_C(a) + \gamma_1^{-1}\gamma_3 n_C(b) + \gamma_2^{-1}\gamma_1 n_C(c)\big),$$

where n_C is the norm form of C. There is a cubic form N defined on $H_3(C, \Gamma)$ given by

$$N(A) = \alpha_1\alpha_2\alpha_3 - \gamma_3^{-1}\gamma_2\alpha_1 n_C(a) - \gamma_1^{-1}\gamma_3\alpha_2 n_C(b) - \gamma_2^{-1}\gamma_1\alpha_3 n_C(c) + t((ca)b),$$

t denoting the trace on C given by $t(x) = x + \bar{x}$. Every element A of $H_3(C, \Gamma)$ satisfies the cubic polynomial equation

$$X^3 - T(A)X^2 + \tfrac{1}{2}\big(T(A)^2 - T(A^2)\big)X - N(A) = 0. \qquad (\bigstar)$$

By an *exceptional Jordan algebra* over R we mean an algebra J such that for a faithfully flat extension S/R, $J \otimes_R S \simeq H_3(C, \Gamma)$ as S-algebras for some C and Γ defined over S, C, and Γ as above. Such an algebra carries a norm N and a trace T which are descents of the norm and trace of $H_3(C, \Gamma)$. In particular, every element of J satisfies the cubic polynomial equation given by ($\bigstar$). In this paper we shall sometimes use the abbreviation Jordan algebras to mean exceptional Jordan algebras.

There is an obvious notion of an exceptional Jordan algebra over any algebraic scheme. If k is a field, the group of automorphisms of exceptional simple Jordan algebras are precisely the algebraic groups of type F_4 over k.

We conclude this section by giving a criterion for the group $G = \operatorname{Aut} J$, J a reduced exceptional simple Jordan algebra over the field k, to be anisotropic.

LEMMA 3.1. *The following are equivalent*

(i) *G is isotropic.*

(ii) *J contains non-zero nilpotent elements.*

(iii) *J is reduced and the rank 24 subform $\langle \gamma_3^{-1}\gamma_2, \gamma_1^{-1}\gamma_3, \gamma_2^{-1}\gamma_1 \rangle \otimes n_C$ of Q is isotropic.*

(iv) *J contains a non-zero element a with $T(a) = Q(a) = N(a) = 0$.*

Proof. (ii) $\Leftrightarrow$ (iv). Any element a of J satisfies

$$X^3 - T(a)X^2 + \tfrac{1}{2}\big(T(a)^2 - T(a^2)\big)X - N(a) = 0.$$

If a is a non-zero nilpotent then clearly $T(a) = Q(a) = N(a) = 0$. Conversely if (iv) holds then we get $a^3 = 0$ so that a is nilpotent. The equivalence of (iii) and (ii) follows by [AJ, Theorem 6; J2, Exercise 1, p. 381]. We now show (i) $\Rightarrow$ (ii). Since G is isotropic, there exists an injection $i: \mathbf{G_m} \hookrightarrow G$ for a k-split $\mathbf{G}_m$. Let J' denote the subspace of J consisting of all trace zero elements of J. Then $\mathbf{G}_m$ acts faithfully on J' and J' can be written as

$$J' = \bigoplus_{i=1}^{26} k.v_i, \qquad k.v_i \text{ being eigenspaces for } \mathbf{G_m}.$$

For any $i, 1 \leq i \leq 26$ and $t \in \mathbf{G}_m$,

$$t(v_i) = \lambda_i(t)v_i$$

for some character λ_i of $\mathbf{G}_m$. Therefore

$$T(v_i) = \lambda_i(t)T(v_i), \qquad Q(v_i) = \lambda_i(t)^2 Q(v_i), \qquad \text{and}$$
$$N(v_i) = \lambda_i(t)^3 N(v_i).$$

So, if for some i and some t, none of $\lambda_i(t), \lambda_i(t)^2, \lambda_i(t)^3$ is 1, then $T(v_i) = Q(V_i) = N(v_i) = 0$, hence v_i is nilpotent. Otherwise for every $t \in \mathbf{G}_m$, $\lambda_i(t)^6 = 1$ and $t^6(v_i) = v_i$ for all i. This contradicts the fact that $\mathbf{G}_m$ is a subgroup of G which acts faithfully on J'.

(ii) $\Rightarrow$ (i). Since J has non-zero nilpotents, by [AJ, Theorem 6], the involution Γ is equivalent to the involution $\langle 1, -1, 1\rangle$ on J. But then $SO(\Gamma)$ is isotropic and since $SO(\Gamma) \hookrightarrow G$ (see 5.1)), G is isotropic. ∎

LEMMA 3.2. *Let J be an exceptional Jordan algebra over a field k of characteristic not equal to* 2 *or* 3. *Let J' be a Jordan subalgebra of dimension* ≥ 12. *Suppose that the restriction of the trace form Q of J to J' is non-degenerate. Then J' is simple and either* $\dim_k J' = 15$ *or* $J' = J$.

Proof. Without loss of generality we assume that k is algebraically closed. Suppose that there exists an ideal I in J' such that $I^2 = 0$. Let $x \in I$. For $y \in J'$, $xy \in I$ and hence $(xy)^2 = 0$, so that $Q(x, y) = T(xy) = 0$. Since the trace form Q restricted to J' is assumed to be non-generate, we have $x = 0$. Therefore, by [J2, Lemma, p. 239], $J' = I_1 \oplus \cdots \oplus I_n$ with each I_j a simple ideal. Since $I_i I_j \subset I_i \cap I_j = 0$ for $i \neq j$, the trace form Q restricted to each I_j is non-degenerate. Let $e_j \in I_j$ be such that $1 = e_1 + \cdots + e_n$. Then each e_j is an idempotent and I_j is a Jordan algebra with e_j as the identity element. Suppose that $\dim_k I_j \geq 2$ for some j. Since k is algebraically closed and Q restricted to I_j is non-degenerate, there exists non-trivial idempotents $e'_j, e''_j \in I_j$ such that $e_j = e'_j + e''_j$. Since $T(e) = 1$

or 2 for any non-trivial idempotent e (cf. [J2, p. 359]) and $T(1) = 3$, it follows that 1 cannot be written as a sum of more than three non-trivial idempotents. Therefore either $n = 1$ or $n = 2$ and one of the two ideals is one dimensional. Suppose that $n = 2$ and $\dim_k I_1 = 1$. Since $\dim_k I_2 \geq 2$, e_2 is a sum of two idempotents and hence e_1 is a primitive idempotent in J; i.e., it cannot be written as a sum of two non-trivial idempotents in J. Let $E = \{x \in J | e_1 x = 0\}$. Then $\dim_k E = 10$ (cf. [J2, p. 367]) and $I_2 \subset E$. Since $\dim_k I_2 = \dim_k J' - 1 \geq 11$, we get a contradiction. Therefore J' is simple and by [S, Theorem 2], $J' \simeq H_3(\mathcal{C}, \Gamma)$ for some composition algebra $\mathcal{C}$. Now the lemma follows from the fact that any such algebra has dimension either 6, 9, 15, or 27. ∎

4. EXTENSIONS OF CAYLEY AND JORDAN ALGEBRAS FROM THE AFFINE PLANE TO THE PROJECTIVE PLANE

Let R be a domain in which 2 and 3 are units. Let C be a Cayley algebra over R and J an exceptional Jordan algebra over R. We call C (resp. J) anisotropic if $C \otimes_R K$ (resp. $J \otimes_R K$) is anisotropic, where K denotes the quotient field of R. We recall that over a field k, C (resp. J) is anisotropic if its group of automorphism $\operatorname{Aut} C$ (resp. $\operatorname{Aut} J$) is anisotropic as an algebraic group over k.

Let $n_C : C \to R$ denote the norm form on C. Then C is anisotropic if and only if n_C is anisotropic. Let $T : J \to R$ be the trace map on the Jordan algebra J, $Q : J \to R$ be the trace form given by $Q(x) = T(x^2)$, and $N : J \to R$ be the norm form. By (3.1) it follows that J is anisotropic if and only if for $x \in J$, $T(x) = Q(x) = N(x) = 0$ implies that $x = 0$. For any integral scheme X and $\mathcal{C}$ a Cayley algebra bundle on X (resp. $\mathcal{J}$ a Jordan algebra bundle), we say that $\mathcal{C}$ is anisotropic (resp. $\mathcal{J}$ is anisotropic) if its restriction to the generic point is anisotropic.

In this section, using methods of [KPS1], we prove some results on extensions of Cayley and Jordan algebra bundles from the affine to the projective plane.

PROPOSITION 4.1. *Let R be as above and $p \in R$ a prime element. Let J and J' be Jordan algebras over R. Suppose that $J' \otimes R/(p)$ is anisotropic. Suppose $\alpha : J \otimes R[1/p] \to J' \otimes R[1/p]$ is an isomorphism of Jordan algebras. Then α extends uniquely to an isomorphism $\tilde{\alpha} : J \to J'$.*

Proof. It is enough to show that $\alpha(J) = J'$. Let $x \in J$. Suppose $\alpha(x) \notin J'$. Let n be the least integer such that $y = p^n\alpha(x) \in J'$ and $p^{n-1}\alpha(x) \notin J'$. Clearly $n \geq 1$. Since α preserves the linear form T, the quadratic form Q, and the cubic form N, we have $T(y) = p^n T(x)$, $Q(y) = p^{2n}Q(x)$, and

$N(y) = p^{3n}N(x)$. Hence $T(y) = Q(y) = N(y) = 0 \mod(p)$ and $y \neq 0 \mod(p)$. This contradicts the assumption that $J' \otimes R/(p)$ is anisotropic. ∎

PROPOSITION 4.2. *Let R and p be as in* (4.1). *Let C, C' be Cayley algebras over R. Suppose that $C' \otimes R/(p)$ is anisotropic. Suppose $\alpha : C \otimes_R R[1/p] \to C' \otimes R[1/p]$ is an isomorphism. Then α extends to an isomorphism $\tilde{\alpha} : C \to C'$.*

Proof. It is enough to show $\alpha(C) = C'$. Let $x \in C$ with $\alpha(x) \notin C'$. Let n be the least integer with $y = p^n\alpha(x) \in C'$ and $p^{n-1}\alpha(x) \notin C'$. Clearly $n \neq 1$. Since α preserves the Cayley norm n_C, we have

$$n_{C'}(y) = p^{2n}n_C(x) = 0 \mod(p) \qquad \text{and} \qquad y \neq 0 \mod(p),$$

contradicting the anisotropicity of $C' \otimes R/(p)$. ∎

COROLLARY 4.3. *Let J, J' be Jordan algebra bundles over $\mathbb{P}^n_k$ such that $\alpha : \mathsf{J} \simeq \mathsf{J}'$ is an isomorphism over $\mathbb{A}^n_k$. Suppose that J and J' are anisotropic. Then α extends uniquely to an isomorphism $\tilde{\alpha}$ of J with J' over $\mathbb{P}^n_k$. The same holds if J and J' are replaced by Cayley algebra bundles C and C'.*

Proof. We give a proof for Jordan algebras which goes through verbatim for Cayley algebras in view of (4.2).

Let $X_0, \ldots, X_n$ be the homogeneous coordinates for $\mathbb{P}^n_k$. Let $D(X_n) = \mathbb{A}^n_k$. Let α_i denote the restriction of α to $D(X_i) \cap D(X_n)$. Since J and J' are anisotropic, by (4.1), α_i extends uniquely to an isometry $\tilde{\alpha}_i$ over $D(X_i)$. Let $\tilde{\alpha}_{ij}$ denote the restriction of $\tilde{\alpha}_i$ to $D(X_i) \cap D(X_j)$. We show that $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}$, so that $\tilde{\alpha} = (\tilde{\alpha}_{ij})$ is an isomorphism over $\mathbb{P}^n_k$. Since $\tilde{\alpha}_{ij}$ and $\tilde{\alpha}_{ji}$ are the restrictions of α to $D(X_i) \cap D(X_j) \cap D(X_n)$, they coincide on this open set. By (4.1), $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji}$. By the very construction, the extension $\tilde{\alpha}$ is unique. ∎

Remark 4.4. The above corollary shows that if C (resp. J) is an anisotropic Cayley (resp. Jordan) algebra over $\mathbb{A}^2_k$ and $\tilde{C}$ (resp. $\tilde{J}$) denotes its extension to $\mathbb{P}^2_k$ then $\mathrm{Aut}_{\mathbb{A}^2_k}C \simeq \mathrm{Aut}_{\mathbb{P}^2_k}\tilde{C}$ (resp. $\mathrm{Aut}_{\mathbb{A}^2_k}J \simeq \mathrm{Aut}_{\mathbb{P}^2_k}\tilde{J}$). See also Section 3.

For the rest of this section we assume that char $k = 0$.

PROPOSITION 4.5. *Let H be a connected reductive algebraic group defined over k. Then every H-bundle over $\mathbb{A}^2_k$ extends to $\mathbb{P}^2_k$ as an H-bundle.*

Proof. Let X, Y, Z be the homogeneous coordinates of $\mathbb{P}^2_k$. Let $x = X/Z$ and $y = Y/Z$. Let E be an H-bundle over $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$ and E_0 its reduction modulo (x, y). In view of [RR, Theorem 1.1], E and E_0 are isomorphic over $k(x)[y]$. Let $f \in k[x]$ be such that $\mathsf{E} \simeq \mathsf{E}_0$ over

Spec$(k[x, y, f^{-1}])$. Let f_Z denote the Z homogenization of f preserving the degree of f. Let $U = D(Z) \cup (D(X) \cap D(f_Z))$. Then the H-bundles $\mathcal{E}$ over $D(Z)$ and $\mathcal{E}_0$ pulled back to $D(X) \cap D(f_Z)$ together with an isomorphism $\alpha : \mathcal{E} \simeq \mathcal{E}_0$ over $D(Z) \cap D(X) \cap D(f_Z)$ define an H-bundle $\tilde{\mathcal{E}}$ over $U = \mathbb{P}^2_k \backslash \{(0:1:0)\}$, which, in turn, extends to an H-bundle over $\mathbb{P}^2_k$ in view of [CTS, Theorem 6.13]. ∎

COROLLARY 4.6. *Every anisotropic Jordan* (*resp. Cayley*) *algebra bundle on* $\mathbb{A}^2_k$ *admits a unique extension to* $\mathbb{P}^2_k$. *If, further, the given bundle on* $\mathbb{A}^2_k$ *admits a reduction of the structure group to a proper connected reductive subgroup of* F_4 *or* G_2 *respectively, its corresponding extension to* $\mathbb{P}^2_k$ *has the same property.*

Proof. This is immediate from (4.3) and (4.5). ∎

We conclude this section with the following proposition which will be used in Section 6.

PROPOSITION 4.7. *Let* $G = \mathrm{Aut}\, J_0$ *be anisotropic and defined over* k. *Let* H *be a connected reductive subgroup of* G. *Let* $\mathcal{J}$ *be a principal* G-*bundle over* $k[X, Y]$ *such that* $\bar{\mathcal{J}} = J_0 \otimes_k R$, *bar denoting modulo* Y, *and* $R = k[Y]$. *Let* $f, g \in R$ *be such that* $(f, g) = 1$. *Let* $\mathcal{J} \otimes R_g[Y] \simeq J_0 \otimes R_g[Y]$. *Suppose* $\mathcal{J} \otimes R_f[Y]$ *has a reduction of the structure group to* H. *Then* $\mathcal{J}$ *has reduction of the structure group to* H.

Proof. The bundle $\mathcal{J}$ is obtained by patching $\mathcal{J} \otimes R_f[Y]$ over $R_f[Y]$ and J_0 over $R_g[Y]$ by an isomorphism $\alpha : \mathcal{J} \otimes R_{fg}[Y] \simeq J_0 \otimes R_{fg}[Y]$. Since $\bar{J} = J_0 \otimes_k R$, one may assume that $\bar{\alpha} =$ identity. The bundle $\mathcal{J} \otimes R_f[Y]$ has a reduction of the structure group to H. Further in view of [RR, Theorem 1.1], $\mathcal{J} \otimes R_f[Y]$ as an H-bundle, is trivial over $k(X)[Y]$. There exists $h \in k[X]$, $(f, h) = 1$, such that $\mathcal{J} \otimes k[X]_{hf}[Y]$ is trivial as an H-bundle, i.e., there exists an isomorphism $\beta : \mathcal{J} \otimes k[X]_{hf}[Y] \simeq J_0 \otimes k[X]_{hf}[Y]$ of H-bundles such that $\bar{\beta} =$ identity. We therefore have $\alpha\beta^{-1} \in \mathrm{Aut}(J_0 \otimes k(X)[Y])$ such that $\overline{\alpha\beta^{-1}} =$ identity. Since J_0 is anisotropic, $\mathrm{Aut}(J_0 \otimes k(X)[Y]) = \mathrm{Aut}(J_0 \otimes k(X))$ (cf. Proposition 5.4), $\alpha\beta^{-1} =$ identity, hence $\alpha = \beta$ over $k(X)[Y]$. Thus the trivialization α, which is an isomorphism of H-bundles generically, is an isomorphism of H-bundles over $k[X]_{fg}[Y]$. Hence $\mathcal{J}$ has a reduction of the structure group to H. ∎

5. AUTOMORPHISMS OF REDUCED EXCEPTIONAL JORDAN ALGEBRAS

Let R be a ring in which 2 and 3 are units. Let C_0 be a Cayley algebra over R. Let $J_0 = \mathcal{H}_3(C_0, \Gamma)$ denote the exceptional Jordan algebra over R defined in Section 3, corresponding to the involution given by the matrix

$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, $\gamma_i \in R^*$. Let $e_1 = \langle 1, 0, 0 \rangle$, $e_2 = \langle 0, 1, 0 \rangle$, and $e_3 = \langle 0, 0, 1 \rangle$ denote the standard idempotents of $H_3(C_0, \Gamma)$. For $x \in C_0$, let

$$X_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \gamma_3^{-1}\gamma_2\bar{x} & 0 \end{pmatrix}, \qquad X_2(x) = \begin{pmatrix} 0 & 0 & \gamma_1^{-1}\gamma_3\bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}$$

and

$$X_3(x) = \begin{pmatrix} 0 & x & 0 \\ \gamma_2^{-1}\gamma_1\bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We denote by X_i the R-submodule of $H_3(C_0, \Gamma)$ consisting of the matrices $X_i(x)$ and by X_i' the submodule of matrices $X_i(x)$ with $t(x) = 0$.

Let $R_{(i)} = \{X_i(\lambda) \mid \lambda \in R\}$. Then $X_i = R_{(i)} \perp X_i'$ and with respect to the trace form, J_0 decomposes as

$$J_0 = Re_1 \perp Re_2 \perp Re_3 \perp X_1 \perp X_2 \perp X_3.$$

The trace form restricted to $Re_1 \perp Re_2 \perp Re_3$ is $\langle 1, 1, 1 \rangle$ and when restricted to $X_1 \perp X_2 \perp X_3$ it is $2(\gamma_3^{-1}\gamma_2 n_0 \perp \gamma_1^{-1}\gamma_3 n_0 \perp \gamma_2^{-1}\gamma_1 n_0)$, where $n_0 = n_{C_0}$ is the norm on the Cayley algebra C_0.

LEMMA 5.1. *Let $A \in O(\Gamma)$. The map $\theta \mapsto A\theta A^{-1}$ is an automorphism of $H_3(C_0, \Gamma)$.*

Proof. Since A has entries in R, the expression $A\theta A^{-1}$ makes sense. For $\theta \in H_3(C_0, \Gamma)$ we have $\theta = \Gamma^{-1}\bar{\theta}^t\Gamma$ and

$$\begin{aligned} \Gamma^{-1}\left(\overline{A\theta A^{-1}}\right)^t \Gamma = \Gamma^{-1}(A\bar{\theta}A^{-1})^t\Gamma &= \Gamma^{-1}\left(A^{t^{-1}}\bar{\theta}^t A^t\right)\Gamma \\ &= A\theta A^{-1}, \end{aligned}$$

since $A \in O(\Gamma)$ and $A^t\Gamma A = \Gamma$. It is easy to verify that this map is a Jordan algebra automorphism. ∎

We denote the above automorphism by Int_A.

LEMMA 5.2. *Let R be a domain. Let*

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in SO(\Gamma)$$

with $a, b, c, d \in R$ such that $abcd \neq 0$. Let $\psi_2 = \mathrm{Int}_B$, $\psi_3 = \mathrm{Int}_A$. Then the following holds.

(i) *ψ_2 stabilizes $X_1 \perp X_3$ and X_2' and takes every non-zero element of X_1 outside X_1 and that of X_3 outside X_3. Further, it fixes the idempotent of e_2 and maps the rank 2 submodule $R_{(2)} \perp R(e_1 - e_3)$ into itself.*

(ii) *ψ_3 stabilizes $X_1 \perp X_2$ and X_3' and takes every non-zero element of X_1 outside X_1 and that of X_2 outside X_2. Further, it fixes the idempotent e_3 and maps the rank 2 submodule $R_{(3)} \perp R(e_1 - e_2)$ into itself.*

Proof. We give a proof of (ii). A proof for (i) follows on similar lines. We have

$$\psi_3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \gamma_3^{-1}\gamma_2\bar{x} & 0 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \gamma_3^{-1}\gamma_2\bar{x} & 0 \end{pmatrix}\begin{pmatrix} d & -b & 0 \\ -c & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & bx \\ 0 & 0 & dx \\ -c\gamma_3^{-1}\gamma_2\bar{x} & a\gamma_3^{-1}\gamma_2\bar{x} & 0 \end{pmatrix},$$

$$\psi_3\begin{pmatrix} 0 & 0 & \gamma_1^{-1}\gamma_3\bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & \gamma_1^{-1}\gamma_3\bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}\begin{pmatrix} d & -b & 0 \\ -c & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & a\gamma_1^{-1}\gamma_3\bar{x} \\ 0 & 0 & c\gamma_1^{-1}\gamma_3\bar{x} \\ dx & -bx & 0 \end{pmatrix}.$$

Thus ψ_3 stabilizes $X_1 \perp X_2$. Further, since $abcd \neq 0$, ψ_3 maps any non-zero element of X_1 outside X_1 and that of X_2 outside X_2. Now,

$$\psi_3\begin{pmatrix} 0 & x & 0 \\ \gamma_2^{-1}\gamma_1\bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} bd\gamma_2^{-1}\gamma_1\bar{x} - acx & -b^2\gamma_2^{-1}\gamma_1\bar{x} + a^2x & 0 \\ d^2\gamma_2^{-1}\gamma_1\bar{x} - c^2x & -bd\gamma_2^{-1}\gamma_1\bar{x} + acx & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $A \in SO(\Gamma)$, $bd\gamma_2^{-1}\gamma_1 + ac = 0$, so that $\psi_3(X_3') = X_3'$. Finally,

$$\psi_3(e_1 - e_2) = \begin{pmatrix} ad + bc & -2ab & 0 \\ 2cd & -(bc + ad) & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R_{(3)} \perp R(e_1 - e_2)$$

and by the computation done before, $\psi_3(X_3 \perp R(e_1 - e_2)) \subset X_3 \perp R(e_1 - e_2)$ and ψ_3 maps $R_{(3)} \perp R(e_1 - e_2)$ into itself. ∎

Recall the Moufang identity in C_0;

$$a(xy)a = (ax)(ya), \qquad a, x, y \in C_0.$$

For $a \in C_0$, we define maps U_a, R_a, L_a of C_0 into itself by

$$U_a(x) = axa, \qquad L_a(x) = ax, \qquad R_a(x) = xa.$$

Then the Moufang identity gives

$$U_a(xy) = L_a(x) R_a(y).$$

For $a \in C_0$ with $n_0(a) = 1$, U_a, L_a, and R_a belong to $O(n_0)$. Choose $a \in C_0 \setminus R$ with $n_0(a) = 1$ with $t(a) \neq 0$. We define $\psi_a : J_0 \to J_0$ as

$$\psi_a(e_i) = e_i, \qquad i = 1,2,3.$$

$$\psi_a(X_3(x)) = X_3(L_a(x)), \quad \psi_a(X_2(x)) = X_2(U_a(x))$$

and

$$\psi_a(X_1(x)) = X_1(R_a(x)).$$

Then ψ_a is an automorphism of J_0 (cf., [J1, p. 419, Proof of Theorem 6]). Further, for $(x, y, z) \in X_1 \perp X_2 \perp X_3$, $\psi_a(x, y, z) = (xa, aya, az)$. Thus ψ_a stabilizes X_1, X_2 and X_3. Also since $n_0(a) = 1$ and $t(a) \neq 0$, $a^2 \notin R$. Hence ψ_a maps $R_{(i)}$ into X_i such that no non-zero element of $R_{(i)}$ is mapped into $R_{(i)}$. We record this as

LEMMA 5.3. *The map $\psi_a : J_0 \to J_0$, defined as above, is an automorphism of J_0 with the property that ψ_a stabilizes each X_i and maps $R_{(i)}$ into X_i such that no non-zero element of $R_{(i)}$ is mapped into $R_{(i)}$.*

We record here some rigidity properties of automorphisms of Jordan and Cayley algebras over polynomial rings. We compare this with a corresponding statement for orthogonal groups in [PS, Lemma 2.2].

PROPOSITION 5.4. *Let J_0 be a Jordan algebra over a domain R, which is anisotropic. Then*

$$\mathrm{Aut}_{R[Y]}(J_0 \otimes R[Y]) = \mathrm{Aut}_R J_0.$$

Proof. Let $\alpha \in \mathrm{Aut}_{R[Y]}(J_0 \otimes R[Y])$. Let $v_0 \in J_0$ and $\alpha(v_0) = w_0 + w_1 Y + \cdots + w_k Y^k$, $w_i \in J_0$. Suppose that $k \geq 1$ and $w_k \neq 0$. Then $N(\alpha(v_0)) = N(w_k)Y^{3k} +$ (a polynomial of degree $< 3k) = N(v_0) \in R$. Hence $N(w_k) = 0$. Also, $T(\alpha(v_0^2)) = T(w_k^2).Y^{2k} +$ (a polynomial of degree $< 2k) = T(v_0^2) \in R$. Thus $T(w_k^2) = 0$. Now, $T(\alpha(v_0)) = T(w_k).Y^k$ + (a polynomial of deg $< k) = T(v_0) \in R$. Hence $T(w_k) = 0$. So $T(w_k) = Q(w_k) = N(w_k) = 0$. Since J_0 is anisotropic, $w_k = 0$. This leads to a contradiction. Thus $\alpha(v_0) \in J_0$ and $\alpha = \alpha_0 \otimes 1$ for $\alpha_0 \in \mathrm{Aut}_R J_0$. This proves the proposition. ∎

PROPOSITION 5.4′. *Let C_0 be an anisotropic Cayley algebra over R. Then* $\mathrm{Aut}_{R[Y]}(C_0 \otimes R[Y]) = \mathrm{Aut}_R C_0$.

Proof. This is immediate from [PS, Lemma 2.2] since n_{C_0} is anisotropic and automorphisms preserve norm. ∎

PROPOSITION 5.5. *Let R be a domain and $p, h \in R$ with $Rp + Rh = R$. Let J_1, J_2 be anisotropic Jordan algebras over $R[Y]$ which are extended after inverting p. Let $\varphi : (J_1)_h \to (J_2)_h$ be an isometry over $R_h[Y]$ such that $\overline{\varphi}$ is defined over R. Then φ is defined over $R[Y]$.*

Proof. The proof goes on parallel lines as that of [P, 2.4], using (5.4). ∎

6. CONSTRUCTION OF NON-TRIVIAL JORDAN ALGEBRAS OVER $k[X, Y]$

We begin with the following "general nonsense" lemma which will be used in the sequel.

LEMMA 6.1. *Let R be a domain and A an algebra (not necessarily associative) over $R[Y]$. Let A_0 be an algebra over R such that A, as an $R[Y]$ module, is isomorphic to $A_0 \otimes R[Y]$ and $\overline{A}$ is isomorphic to A_0 as an R-algebra, bar denoting modulo Y. Then there exists an algebra structure A' on $A_0 \otimes_R R[Y]$, isomorphic to A, such that $\overline{A'} = A_0$. (Here, the natural map $\overline{A_0 \otimes_R R[Y]} \simeq A_0$ is treated as an identification.)*

Proof. Let $\overline{\beta} : A_0 \to \overline{A}$ be an isomorphism of algebras over R. Let $\alpha : A_0 \otimes_R R[Y] \simeq A$ be an isomorphism of $R[Y]$-modules and $\overline{\alpha} : A_0 \to \overline{A}$ its reduction modulo Y. Let $\beta = \alpha \circ ((\overline{\alpha}^{-1} \circ \overline{\beta}) \otimes 1)$. We pull back the algebra structure on A to an algebra structure on $A' = A_0 \otimes R[Y]$ through $\beta : A_0 \otimes R[Y] \to A$, i.e., we set, for $x, y \in A'$

$$x * y = \beta^{-1}(\beta(x).\beta(y)).$$

Then $\beta : A' \to A$ is an isomorphism of $(A', *)$ with $(A, .)$ Further,

$$\bar{x} * \bar{y} = \overline{\beta}^{-1}\big(\overline{\beta}(x).\overline{\beta}(y)\big) = \overline{\beta}^{-1}\big(\overline{\beta}(\bar{x}.\bar{y})\big) = \bar{x} \circ \bar{y},$$

where $\bar{x} \circ \bar{y}$ denotes multiplication in A_0. Thus

$$\overline{A'} = A_0.$$ ∎

Let k be a field of characteristic not equal to 2 and 3, admitting a reduced anisotropic Jordan algebra $J_0 = H_3(C_0, \Gamma)$, where C_0 is a Cayley division algebra over k and $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, $\gamma_i \in k^*$. Let $G = \mathrm{Aut}\, J_0$. In this section, we construct, in several steps, a family of Jordan algebras $J^{[i]}$

over $k[X,Y]$ which are rigid in the following sense: the principal G-bundle associated to $J^{[i]}$ over $\mathbb{A}_k^2$ does not admit reduction of the structure group to any proper connected reductive subgroup of G. This will be shown in Section 7. We begin with the construction.

Step I. By [T, Theorem 3.1], there exist infinitely many pairs $(C_i, p_i)_{i\geq 1}$ of mutually non-isomorphic Cayley algebras C_i over $k[X,Y]$ and polynomials $p_i \in k[X]$ with $(p_i, p_j) = 1$ for $i \neq j$, with the following properties: if bar denotes reduction modulo Y, $\overline{C}_i \simeq C_0 \otimes k[Y]$, $C_i \otimes k[X]_{p_i}[Y] \simeq C_0 \otimes k[X]_{p_i}[Y]$ and the rank 7-quadratic space C_i' of trace zero elements of C_i is indecomposable over $k[X,Y]$. Let n' denote the norm on the Cayley algebra restricted to trace zero elements. Twisting C_i through an isomorphism $\beta_i : \overline{C}_i \simeq C_0 \otimes k[Y]$, as in (6.1), we may and do assume that $\overline{C}_i = C_0 \otimes k[Y]$. Let $\pi_i : C_i \otimes k[X]_{p_i}[Y] \simeq C_0 \otimes k[X]_{p_i}[Y]$ be isomorphism such that $\overline{\pi}_i =$ identity. Let $J_i = H_3(C_i, \Gamma)$, $i \geq 1$. Then $\bar{J}_i = J_0 = H_3(C_0, \Gamma)$ and π_i gives rise to the isomorphism

$$\pi_i : J_i \otimes k[X]_{p_i}[Y] \simeq J_0 \otimes k[X]_{p_i}[Y]$$

of Jordan algebras with $\overline{\pi}_i =$ identity.

PROPOSITION 6.2. *The algebras J_i are mutually non-isomorphic.*

Proof. Suppose $J_i \simeq J_j$ for $i \neq j$. Since J_i is extended from J_0 over $k[X]_{p_i}[Y]$ and J_j from J_0 over $k[X]_{p_j}[Y]$ with $(p_i, p_j) = 1$, J_i on $k[X][Y]$, is locally extended from $k[X]$ and by [BCW, 4.15], $J_i \simeq J_0 \otimes k[X,Y]$. Let $\tilde{C}_i$ be an extension of the Cayley algebra C_i to $\mathbb{P}_k^2$. Then $\tilde{J}_i = H_3(\tilde{C}_i, \Gamma)$ is the extension of J_i to $\mathbb{P}_k^2$, by the unique extension property (4.6). The underlying vector bundle of $\tilde{J}_i$ decomposes as

$$\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}e_3 \oplus \left(\tilde{X}_1' \oplus \mathcal{O}\right) \oplus \left(\tilde{X}_2' \oplus \mathcal{O}\right) \oplus \left(\tilde{X}_3' \oplus \mathcal{O}\right),$$

where $\mathcal{O}$ denotes the structure sheaf of $\mathbb{P}_k^2$ and $\tilde{X}_j'$ the trace zero elements in $\tilde{C}_i$ sitting in 3 copies of $\tilde{C}_i$ in $\tilde{J}_i$. The bundles $\tilde{X}_j'$ are absolutely indecomposable (Proposition 2.3, Corollary 2.4). Hence $\tilde{J}_i$ is non-trivial as a vector bundle. On the other hand, $J_i \simeq J_0 \otimes k[X,Y]$ and $\pi^* J_0 \simeq \tilde{J}_i$, where $\pi : \mathbb{P}_k^2 \to \operatorname{Spec} k$ is the structure morphism, by the uniqueness of the extension. However, $\pi^* J_0$ is trivial as a vector bundle, leading to a contradiction. This proves the proposition. ∎

Step II. In this step, we patch the bundles J_i locally to get new bundles on $\mathbb{A}_k^2$. More precisely, we take the algebra $(J_{2i-1})_{p_{2i}}$ over $k[X]_{p_{2i}}[Y]$ and $(J_{2i})_{p_{2i-1}}$ over $k[X]_{p_{2i}}[Y]$ and patch them over $k[X]_{p_{2i-1}p_{2i}}[Y]$ through an isomorphism

$$\theta_1^{(i)} : (J_{2i-1})_{p_{2i-1}p_{2i}} \simeq (J_{2i})_{p_{2i-1}p_{2i}}$$

defined as follows.

Let $a \in C_0 \setminus k$ be an element with $n_0(a) = 1$ and $t(a) \neq 0$. Let $\psi_a : J_0 \to J_0$ be the automorphism defined in (5.3). We set $\theta_1^{(i)} = \pi_{2i}^{-1} \circ \psi_a \circ \pi_{2i-1}$. Let J^i denote the resulting Jordan algebra. Let $q_i = p_{2i-1} p_{2i}$. Then $(q_i, q_j) = 1$ for $i \neq j$ and by the very construction, $(J^i)_{q_i} \simeq J_0 \otimes k[X]_{q_i}[Y]$. Writing $J_i = H_3(C_i, \Gamma)$ as $Re_1 \oplus Re_2 \oplus Re_3 \oplus X_{1,i} \oplus X_{2,i} \oplus X_{3,i}$ (cf. Section 5), with $R = k[X, Y]$, the map $\theta_1^{(i)}$ patches the rank 8 submodule $X_{j,2i-1}$ of J_{2i-1} with $X_{j,2i}$ of J_{2i} to yield a rank 8-submodule X_j^i of J^i for $j = 1, 2, 3$. Further, it patches the idempotents e_j of J_{2i-1} with the corresponding e_j of J_{2i}. Thus J^i decomposes as

$$J^i = Re_1 \oplus Re_2 \oplus Re_3 \oplus X_1^i \oplus X_2^i \oplus X_3^i.$$

PROPOSITION 6.3. *The quadratic spaces X_j^i, for the trace form, are rigid. Further, X_j^i is not similar to X_k^i for $j \neq k$, $j, k = 1, 2, 3$.*

Proof. The space X_j^i is obtained by patching the quadratic space $\lambda_j n_{C_{2i-1}} = \lambda_j(1 \perp n'_{C_{2i-1}})$ on $k[X]_{p_{2i}}[Y]$ and $n_{C_{2i}} = \lambda_j(1 \perp n'_{C_{2i}})$ on $k[X]_{p_{2i-1}}[Y]$ by isometries $\pi_{2i}^{-1} \circ R_a \circ \pi_{2i-1}, \pi_{2i}^{-1} \circ U_a \circ \pi_{2i-1}$ and $\pi_{2i}^{-1} \circ L_a \circ \pi_{2i-1}$, respectively, with $\lambda_j \in k^*$ for $j = 1, 2, 3$. The rank 7 spaces n'_{C_j} are indecomposable over $k[X, Y]$ and hence rigid (2.4). Further, since $t(a) \neq 0, R_a, U_a, L_a : C_0 \to C_0$ map 1 away from 1. Hence by (2.6), the rank 8 space X_j^i is rigid for $j = 1, 2, 3$.

We now show that X_j^i is not similar to X_k^i for $j \neq k$. We do this for $j = 1$, $k = 2$. Other cases follow on similar lines. Suppose $X_1^i \simeq \lambda X_2^i$ for some $\lambda \in k^*$. Then there exist isometries $\varphi_{2i} : 1 \perp n'_{c_{2i-1}} \to \lambda(1 \perp n'_{c_{2i-1}})$ over $k[X]_{p_{2i}}[Y]$ and $\varphi_{2i-1} : 1 \perp n'_{C_{2i}} \to \lambda(1 \perp n'_{C_{2i}})$ over $k[X]_{p_{2i-1}}[Y]$, such that the diagram

$$\begin{array}{ccc} 1 \perp n'_{c_{2i-1}} & \xrightarrow{\varphi_{2i}} & \lambda(1 \perp n'_{c_{2i-1}}) \\ {\scriptstyle \pi_{2i}^{-1} \circ R_a \circ \pi_{2i-1}} \downarrow & & \downarrow {\scriptstyle \pi_{2i}^{-1} \circ U_a \circ \pi_{2i-1}} \\ 1 \perp n'_{C_{2i}} & \xrightarrow{\varphi_{2i-1}} & \lambda(1 \perp n'_{C_{2i}}) \end{array}$$

is commutative.

We have, by reducing modulo $Y, U_a \circ \overline{\varphi}_{2i} = \overline{\varphi}_{2i-1} \circ R_a$. Since U_a and R_a are defined over k and $(p_{2i}, p_{2i-1}) = 1$, it follows [P, 2.3] that φ_{2i} and φ_{2i-1} both extend to isometries of the corresponding spaces over $k[X, Y]$. Since $n'_{C_{2i-1}}$ and $n'_{C_{2i}}$ are both rigid, in view of (2.5), φ_{2i} and φ_{2i-1} break up into isometries of $\langle 1 \rangle \simeq \langle \lambda \rangle$ and $n'_{C_{2i-1}} \simeq \lambda n'_{C_{2i-1}}$ and $n'_{C_{2i}} \simeq \lambda n'_{C_{2i}}$, respectively. It follows that λ is a square and, after scaling, we may assume $\lambda = 1$. For $\gamma + x \in C_{2i} = R.1 \perp C'_{2i}$ we have

$$\varphi_{2i}(\gamma + x) = \epsilon_1 \gamma + \epsilon_2 x$$

and for $\gamma + x \in C_{2i-1} = R.1 \perp C_{2i-1}$

$$\varphi_{2i-1}(\gamma + x) = \epsilon_3\gamma + \epsilon_4 x$$

with $\epsilon_i = \pm 1$ for $1 \leq i \leq 4$. Commmutativity of the above diagram modulo Y shows that if $a = \gamma_0 + x_0$, $U_a \circ \overline{\varphi}_{2i}(1) = \pm a^2$, $\overline{\varphi}_{2i-1} \circ R_a(1) = \epsilon_3\gamma_0 + \epsilon_4 x_0 = \pm a^2$. This contradicts the choice of a such that $t(a) \neq 0$ and $n_{C_0}(a) = 1$. This proves the proposition. ∎

PROPOSITION 6.4. *The algebras J^i, J^j are non-isomorphic for $i \neq j$.*

Proof. Suppose $J^i \simeq J^j$ for $i \neq j$. Then J^i would be extended from J_0 after inverting q_i and q_j, which are coprime. In vew of [BCW, 4.15], $J^i \simeq J_0 \otimes k[X, Y]$. This implies that $J_{2i} \otimes k[X]_{p_{2i-1}}[Y] \simeq J^i \otimes k[X]_{p_{2i-1}}[Y] \simeq J_0 \otimes k[X]_{p_{2i-1}}[Y]$. However, $J_{2i} \otimes k[X]_{p_{2i}}[Y] \simeq J_0 \otimes k[X]_{p_{2i}}[Y]$, so that by [BCW, 4.15], $J_{2i} \simeq J_0 \otimes k[X, Y]$, contradicting (6.2). This proves the proposition. ∎

We replace J^i by a suitable twist (cf. Lemma 6.1), and assume that $\overline{J^i} = J_0$.

Step III. We take $(J^{2i-1})_{q_{2i}}$ over $k[X]_{q_{2i}}[Y]$ and $(J^{2i})_{q_{2i-1}}$ over $k[X]_{q_{2i-1}}[Y]$ and patch them over $k[X]_{q_{2i-1}q_{2i}}[Y]$ by an isomorphism

$$\theta_2^{(i)} : (J^{2i-1})_{q_{2i-1}q_{2i}} \to (J^{2i})_{q_{2i-1}q_{2i}}$$

defined as follows.

Let $\pi^i : J^i \otimes k[X]_{q_i}[Y] \simeq J_0 \otimes k[X]_{q_i}[Y]$ be an isomorphism such that $\overline{\pi^i} =$ identity. Let $\psi_3 : J_0 \to J_0$ be the automorphism defined in (5.2).

We set $\theta_2^{(i)} = (\pi^{2i})^{-1} \circ \psi_3 \circ (\pi^{2i-1})$. Let $J^{(i)}$ denote the resulting Jordan algebra. In a similar fashion, we construct the Jordan algebra $J^{\{i\}}$ through the patching $\theta_3^{(i)}$ which replaces $\theta_2^{(i)}$ above, defined by

$$\theta_3^{(i)} = (\pi^{2i})^{-1} \circ \psi_2 \circ (\pi^{2i-1}).$$

Let $r_i = q_{2i-1}q_{2i}$. Then $(r_i, r_j) = 1$ for $i \neq j$. By the construction, $(J^{(i)})_{r_i} \simeq J_0 \otimes k[X]_{r_i}[Y]$.

The map $\theta_2^{(i)}$ patches the rank 16 subspace $X_1^{2i-1} \perp X_2^{2i-1}$ of J^{2i-1} over $k[X]_{q_{2i}}[Y]$ and $X_1^{2i} \perp X_2^{2i}$ of J^{2i} over $k[X]_{q_{2i-1}}[Y]$ to yield a subspace X_{12}^i of rank 16 in $J^{(i)}$. Further, it patches the subspace $X_3^{2i-1} \perp R(e_1 - e_2)$ of J^{2i-1} with $X_3^{2i} \perp R(e_1 - e_2)$ of J^{2i} to yield a subspace $X_{(3,1/2)}^i$ of $J^{(i)}$ of rank 9. Since ψ_3 fixes $e_1 + e_2 - 2e_3$, $J^{(i)}$ has a rank 1 trivial summand generated by $e_1 + e_2 - 2e_3$. Thus $J^{(i)}$ decomposes as an orthogonal bundle as

$$J^{(i)} = X_{12}^i \perp X_{(3,1/2)}^i \perp R(e_1 + e_2 - 2e_3) \perp R(e_1 + e_2 + e_3).$$

The bundle $J^{\{i\}}$ has a similar decomposition

$$J^{\{i\}} = X_{13}^i \perp X_{(2,1/3)}^i \perp R(e_1 + e_3 - 2e_2) \perp R(e_1 + e_2 + e_3).$$

PROPOSITION 6.5. *The subspaces X_{12}^i and $X_{(3,1/2)}^i$ of $J^{(i)}$ (resp. X_{13}^i and $X_{(2,1/3)}^i$ of $J^{\{i\}}$ are rigid and hence indecomposable.*

Proof. The space X_{12}^i is obtained by patching the sum $X_1^{2i-1} \perp X_2^{2i-1}$ of rigid subspaces of J^{2i} through $\theta_2^{(i)}$ with $\bar{\theta}_2^{(i)} = \psi_3$. The map ψ_3 maps every non-zero element of X_1 outside X_1 and that of X_2 outside X_2. Further, X_1^j is not similar to X_2^j for any j (6.3). In view of (2.6), it follows that X_{12}^i is rigid. Since $\theta_2^{(i)}$ patches the subspace $X_3^{2i-1} \perp R(e_1 - e_2)$ of J^{2i-1} with $X_3^{2i} \perp R(e_1 - e_2)$ of J^{2i} with $\bar{\theta}_2^{(i)} = \psi_3$ mapping $k(e_1 - e_2)$ outside $k(e_1 - e_2)$ and $\psi_3(X_3) \not\subset X_3$, in view of (2.6), $X_{(3,1/2)}^i$ is rigid. A similar argument yields the proposition for $J^{\{i\}}$ as well.

PROPOSITION 6.6. *The algebras $J^{(i)}, J^{(j)}$ are non-isomorphic for $i \neq j$. Similarly, the algebras $J^{\{i\}}$ and $J^{\{i\}}$ are non-isomorphic for $i \neq j$.*

Proof. Suppose $J^{(i)} \simeq J^{(j)}$, $i \neq j$. Arguing as in the proof of (6.4), $J^{(i)}$ is isomorphic to J_0 over $k[X]_{r_i}[Y]$ as well as $k[X]_{r_j}[Y]$ and hence is isomorphic to $J_0 \otimes k[X,Y]$ [BCW, 4.15]. Hence $J^{2i-1} \otimes k[X]_{q_{2i}}[Y] \simeq J^{(i)} \otimes k[X]_{q_{2i}}[Y] \simeq J_0 \otimes k[X]_{q_{2i}}[Y]$. Further, $J^{2i-1} \otimes k[X]_{q_{2i-1}}[Y] \simeq J_0 \otimes k[X]_{q_{2i-1}}[Y]$ and this would imply that $J^{2i-1} \simeq J_0 \otimes k[X,Y]$, contradicting (6.4). That the algebras $J^{\{i\}}, J^{\{j\}}$ are non-isomorphic for $i \neq j$ follows on similar lines. ∎

We replace $J^{(i)}$ and $J^{\{i\}}$ by a suitable twist (6.1) and assume, without loss of generality, that $\overline{J^{(i)}} = J_0 = \overline{J^{\{i\}}}$.

Step IV. We take the space $J_{r_{i+1}}^{(i)}$ over $k[X]_{r_{i+1}}[Y]$ and $J_{r_i}^{\{i+1\}}$ over $k[X]_{r_i}[Y]$ and patch them over $k[X]_{r_i r_{i+1}}[Y]$ by an isometry

$$\theta_4^{(i)} : J^{(i)} \otimes k[X]_{r_i r_{i+1}}[Y] \to J^{\{i+1\}} \otimes k[X]_{r_i r_{i+1}}[Y]$$

defined as follows. Let

$$\pi_{(i)} : J^{(i)} \otimes k[X]_{r_i}[Y] \to J_0 \otimes k[X]_{r_i}[Y]$$

$$\pi_{\{i\}} : J^{\{i\}} \otimes k[X]_{r_i}[Y] \to J_0 \otimes k[X]_{r_i}[Y]$$

be isomorphisms with $\bar{\pi}_{(i)} = \text{identity}, \bar{\pi}_{\{i\}} = \text{identity}$. We set $\theta_4^{(i)} = \pi_{\{i+1\}}^{-1} \pi_{(i)}$. Let $J^{[i]}$ be the resulting Jordan algebra over $k[X,Y]$. Let $s_i = r_i r_{i+1}$. Then $(s_i, s_j) = 1$ for $i \neq j$ and by construction, $(j^{[i]})_{s_i} \simeq J_0 \otimes K[X]_{s_i}[Y]$.

PROPOSITION 6.7. *The algebras $J^{[i]}$ and $J^{[j]}$ are non-isomorphic for $i \neq j$.*

Proof. The proof runs on exactly the same lines as (6.4) noting that $J^{[i]} \otimes k[X]_{s_i}[Y] \simeq J_0 \otimes k[X]_{s_i}[Y]$ for all i and using (6.5). ∎

We shall show in the next section that the algebras $J^{[i]}$ are rigid in the following sense. If $G = \mathrm{Aut}\, J_0$ and $J^{[i]\prime}$ denotes the trace zero elements in $J^{[i]}$, $J^{[i]\prime}$ as a G-bundle, admits no reduction of the structure group to any proper connected reductive subgroup of G.

7. F_4 BUNDLES WITH NO REDUCTION OF THE STRUCTURE GROUP TO PROPER CONNECTED REDUCTIVE SUBGROUPS

In this section we prove that the principal G-bundles associated to the Jordan algebras $J^{[i]}$ over $\mathbb{A}_k^2$, constructed in Section 6, have no reduction of the structure group to any proper connected reductive subgroup.

We begin with some lemmas on representations of algebraic groups.

LEMMA 7.1. *Let G be an algebraic group over a field k. Let V be a representation of G over k. Suppose that $V \otimes \bar{k} \simeq (W \oplus \cdots \oplus W) \otimes_k \bar{k}$ for some representation W of G which is irreducible over the algebraic closure $\bar{k}$ of k. Then $V \simeq W \oplus \cdots \oplus W$.*

Proof. Since $V \otimes_k \bar{k} \simeq (W \oplus \cdots \oplus W) \otimes_k \bar{k}$, it corresponds to an element in $H^1(k, \mathrm{Aut}_G((W \oplus \cdots \oplus W) \otimes_k \bar{k}))$. Since $W \otimes_k \bar{k}$ is an irreducible representation of G, we have $H^1(k, \mathrm{Aut}_G((W \oplus \cdots \oplus W) \otimes_k \bar{k})) \simeq H^1(k, GL_n) = 0$. Therefore $V \simeq W \oplus \cdots \oplus W$. ∎

We, now on, assume that char $k = 0$.

LEMMA 7.2. *Let H be a simply connected, simple algebraic group over k of type A_1. Let V_0 be the 2 dimensional representation of H over $\bar{k}$. Then the symmetric power $\mathrm{Sym}^{2n}(V_0)$ of V_0 is defined over k for all n.*

Proof. If H is isotropic over k then V_0 is defined over k. So assume H is anisotropic over k. Then $H \simeq SL(D)$ for some quaternion division algebra D over k and $\mathrm{Sym}^2(V_0) \simeq D_0 \otimes \bar{k}$ where D_0 is the space of trace zero elements of D. Thus $\mathrm{Sym}^2(V_0)$ is defined over k. We prove the lemma by induction on n. Assume that $\mathrm{Sym}^{2i}(V_0)$ is defined over k for $1 \leq i \leq n-1$. We have

$$\mathrm{Sym}^{2(n-1)}(V_0) \otimes \mathrm{Sym}^2(V_0) \simeq \mathrm{Sym}^{2n}(V_0) \oplus \mathrm{Sym}^{2(n-1)}(V_0) \oplus \cdots \oplus \mathrm{Sym}^0(V_0)$$

(cf. [OV, p. 300]). By induction, $\mathrm{Sym}^{2i}(V_0)$ is defined over k for $1 \leq i \leq n - 1$, thus $\mathrm{Sym}^{2(n-1)}(V_0) \otimes \mathrm{Sym}^2(V_0)$ is defined over k and hence $\mathrm{Sym}^{2n}(V_0)$ is defined over k. ∎

LEMMA 7.3. *Let H be as in* (7.2). *Suppose further that H is anisotropic. Let V be a faithful representation of H of dimension $2p$, with p an odd prime. If $V \otimes \bar{k}$ is reducible, then V is reducible.*

Proof. Suppose that $V \otimes \bar{k}$ is reducible and V is irreducible. Then $V \otimes \bar{k} \simeq W \oplus W$, W being the p-dimensional irreducible representation of H or $V \otimes \bar{k} \simeq V_0 \oplus \cdots \oplus V_0, V_0$ being the 2-dimensional irreducible representation of H. Since $W \simeq \mathrm{Sym}^{p-1}(V_0)$, p being odd, it is defined over k (7.2). Thus if $V \otimes \bar{k} \simeq W \oplus W$, then by (7.1), V is reducible, contradicting the assumption. Suppose that $V \otimes \bar{k} \simeq V_0 \oplus \cdots \oplus V_0$. Since V_0 is defined over a quadratic extension L of k by (7.1), $V \otimes_k L \simeq V_0 \oplus \cdots \oplus V_0$. We note that the non-trivial automorphism σ of L over k permutes the V_0 summands. Since p is odd and there are p copies of V_0 in $V \otimes_k L$, it follows that there is a summand V_0 which is σ-stable, contradicting the assumption on H. This completes the proof. ∎

LEMMA 7.4. *Let H be a semisimple algebraic group over an algebraically closed field and V a finite dimensional representation of H. Suppose that there is a non-singular quadratic form q on V which is H-invariant. If $V = W_1 \oplus W_2$, where W_2 is the isotypical component for the trivial representation and W_1 is H-stable, then the restriction of q to W_1 is non-degenerate.*

Proof. Let b be the bilinear form associated to q. Let $U = \{x \in W_1 \mid b(x, y) = 0 \text{ for all } y \in W_1\}$ be the radical of q. Since q is H-invariant, H stabilizes U. Suppose that $U \neq 0$. Let U' be an irreducible subrepresentation of U. By the assumption on W_1, $\dim_k U' \geq 2$. Let H' be a simple factor of H such that H' acts non-trivially on U'. Let T be a maximal torus of H'. Since k is algebraically closed, the action of T on U' is diagonalisable. Let $x \in U'$ be an eigenvector with eigencharacter χ for T. Since q is non-singular on V, there exists $y \in V$ such that $b(x, y) = 1$. Since $b(x, w) = 0$ for all $w \in W_1$, we assume that $y \in W_2$. For any $t \in T$, we have

$$1 = b(x, y) = b(tx, ty) = b(\chi(t)x, y) = \chi(t)b(x, y) = \chi(t).$$

Thus the action of T on U' is trivial and hence H' acts trivially on U', a contradiction. ∎

Let G be a connected reductive group over k and V a finite dimensional representation of G. We say that V is of *type n* if V is a sum of irreducible representations of dimension n. Suppose that V is irreducible. Then it is easily verified that $V_{\bar{k}} = V \otimes_k \bar{k}$ is of type n, for some n.

Let C_0 be a Cayley division algebra over k. Let $\Gamma = \langle \gamma_1, \gamma_2, \gamma_2 \rangle$, $\gamma_i \in k^*$ and $J_0 = H_3(C_0, \Gamma)$. Let $G = \operatorname{Aut} J_0$. Let V be the subspace of trace zero elements in J_0.

LEMMA 7.5. *Let H be a proper connected reductive subgroup of G. Then H acts reducibly on V.*

Proof. Replacing H by the commutator subgroup of H, we assume that H is semisimple. Suppose that the action of H on V is irreducible. Then $V_{\bar{k}}$ as a H-representation is of type n, for some n. Since the action of $H_{\bar{k}} = H \times \bar{k}$ on $V_{\bar{k}}$ is reducible [D, Theorem 16.1, p. 239] and $\dim_k(V) = 26$, for the action of $H_{\bar{k}}, V_{\bar{k}}$ decomposes as $\oplus W_i$ with W_i irreducible and $\dim_k W_i = 13$, $i = 1, 2$ or $\dim_k(W_i) = 2$, $1 \le i \le 13$. Let H_1 be a simple factor of $H_{\bar{k}}$. Since the action of H on $V_{\bar{k}}$ is faithful, there exists an i such that H_1 acts faithfully on W_i. Since $\dim_k W_i = 2$ or 13, it follows, from the table of formulae for dimensions of irreducible representations of simple groups over an algebraically closed field [OV, pp. 300−305], that H is of type A_1. By (7.3), $H_{\bar{k}}$ has more than one simple component. We write $H_{\bar{k}}$ as an almost direct product $H_1 H_2$, with H_1 simple. We observe that every irreducible representation of $H_1 H_2$ decomposes as $V_1 \otimes V_2$, where V_i is an irreducible H_i-representation, for $i = 1, 2$. Since each W_i is an irreducible $H_{\bar{k}}$-representation of dimension a prime, the action of H_1 on each W_i is either irreducible or trivial. Further, if the action of H_1 is irreducible on W_i for some i, then the action of H_2 on W_i is trivial. Since the action of H on V is faithful, each H_i acts non-trivially on some W_j. Let us first consider the case where $V_{\bar{k}} = W_1 \oplus W_2$, with $\dim_k W_i = 13$, $i = 1, 2$. Then H_2 is also simple. Without loss of generality we assume that H_1 acts irreducibly on W_1. Then H_2 acts trivially on W_1 and irreducibly on W_2. Therefore, by (7.4), the restriction of the trace form of J to W_1 is non-degenerate. Let J_1 be the Jordan subalgebra of J generated by W_1. Since H_2 acts trivially on W_1, it acts trivially on J_1. Suppose that $k \oplus W_1$ is a proper subspace of J_1. Then $\dim_k J_1 \ge 15$ and $J_1 \cap W_2 \ne 0$ is a trivial subrepresentation of W_2 for H_2, leading to a contradiction. Hence $J_1 = k \oplus W_1$. Since $\dim_k J_1 = 14$ and the trace form restricted to $k \oplus W_1$ is non-degenerate, this once again leads to a contradiction in view of (3.2). We may therefore assume that $V_{\bar{k}} = W_1 \oplus \cdots \oplus W_{13}$, with $\dim_k W_i = 2$, $1 \le i \le 13$. Let r, $1 \le r \le 13$, be such that H_1 acts irreducibly on W_i for $1 \le i \le r$ and trivially on W_j for $r + 1 \le j \le 13$. Then H_2 acts trivially on W_i for $1 \le i \le r$. By (7.4), the trace form restricted to $W_{r+1} \oplus \cdots \oplus W_{13}$ is non-degenerate. Since H has at least two simple factors, without loss of generality we assume that $r \le 6$. Let J_2 be the Jordan subalgebra of J generated by $W_{r+1}, \ldots, W_{13}$. Then H_1 acts trivially on J_2. Since $r \le 6$, we have $\dim_k J_2 \ge 14$. Arguing as above and using (3.2), (7.4), we get $(W_1$

$\oplus \cdots \oplus W_r) \cap J_2 \neq 0$. Since H_1 acts irreducibly on W_i for $1 \leq i \leq r$ and trivially on J_2, this is a contradiction. This proves the lemma. ▮

PROPOSITION 7.6. *The Jordan algebra $J^{(i)}$ (resp. $J^{\{i\}}$) admits an extension $\tilde{J}^{(i)}$ (resp. $\tilde{J}^{\{i\}}$) to the projective plane $\mathbb{P}^2_k$ whose underlying bundle has a decomposition*

$$\tilde{J}^{(i)} = \tilde{X}^i_{12} \perp \tilde{X}^i_{(3,1/2)} \perp \mathcal{O}(e_1 + e_2 - 2e_3) \perp \mathcal{O}(e_1 + e_2 + e_3),$$

where $\tilde{X}^i_{12}$ extends the orthogonal bundle X^i_{12}, $\tilde{X}^i_{(3,1/2)}$ extends the orthogonal bundle $X^i_{(3,1/2)}$. Further, the bundles $\tilde{X}^i_{12}$ and $\tilde{X}^i_{(3,1/2)}$ are absolutely indecomposable. A similar claim holds for $J^{\{i\}}$ with

$$\tilde{J}^{\{i\}} = \tilde{X}^i_{13} \perp \tilde{X}^i_{(2,1/3)} \perp \mathcal{O}(e_1 + e_3 - 2e_2) \perp \mathcal{O}(e_1 + e_2 + e_3).$$

Proof. The bundle $J^{(i)}$ admits reduction of the structure group G to the connected reductive subgroup H which fixes the idempotent e_3 (in fact $H \simeq \mathrm{Spin}_9$). Thus by (4.6), $J^{(i)}$ admits a unique extension $\tilde{J}^{(i)}$ to $\mathbb{P}^2_k$ which has a reduction of the structure group to H. Since H maps $X_1 \perp X_2$ into itself, $X_3 \perp k(e_1 - e_2)$ into itself and fixes $e_1 + e_2 - 2e_3$, $\tilde{J}^{(i)}$ has a decomposition as given in the proposition. Further, since X^i_{12} and $X^i_{(3,1/2)}$ are rigid (6.5), it follows that $\tilde{X}_{12}$ and $\tilde{X}^i_{(3,1/2)}$ are absolutely indecomposable. Proof of the claims for $J^{\{i\}}$ follows on similar lines. ▮

An exceptional Jordan algebra $\mathbf{J}$ over $\mathbb{A}^2_k$, whose specialization at a rational point is J, gives rise to an etale 1-cocycle $\mathbf{E_J}$ with values in $\mathrm{Aut}\, J = G$. To $\mathbf{E_J}$ is associated a principal G-bundle $P_{\mathbf{J}}$ on $\mathbb{A}^2_k$. Let V be the space of trace zero elements in J. For the faithful representation $G \hookrightarrow \mathrm{Aut}\, V = GL_{26}$, the vector bundle associated to $P_{\mathbf{J}}$ is simply the sub-bundle $\mathbf{J}_0$ of trace zero elements of $\mathbf{J}$.

PROPOSITION 7.7. *Let $J^{[i]}$ be the Jordan algebra constructed in Section 6. Then, the principal G-bundle $P_{J^{[i]}}$ admits no reduction of the structure group to any connected reductive subgroup $H \subset (GL(V_1) \times GL(V_2)) \cap G$, where V_i are non-zero subspaces of V with $V = V_1 \oplus V_2$.*

Proof. Suppose the G-bundle $P_{J^{[i]}}$ has a reduction of the structure group to a connected reductive subgroup $H \subset (GL(V_1) \times GL(V_2)) \cap G$. Then $((J^{[i]})_0)_{r_{i+1}} \simeq ((J^{(i)})_0)_{r_{i+1}}$ over $k[X]_{r_{i+1}}[Y]$ has a reduction of the structure group to H. In view of (4.7), $P_{J^{(i)}}$ over $k[X,Y]$ itself has a reduction of the structure group to H. Let $\tilde{J}^{(i)}$ denote the extension of the Jordan algebra $J^{(i)}$ to $\mathbb{P}^2_k$. The $\tilde{J}^{(i)}$ also admits a reduction of the structure group to H (4.6) and hence to $GL(V_1) \times GL(V_2)$. Hence $(\tilde{J}^{(i)})_0$ decomposes as $\mathbf{E}_1 \oplus \mathbf{E}_2$ with $\mathbf{E}_j$ a $GL(V_j)$-bundle, $j = 1, 2$. We assume without loss of generality that $\dim V_1 \leq 13$. By the uniqueness of extension (4.6) and in view of (7.6),

$$(\tilde{J}^{(i)})_0 = \tilde{X}^i_{12} \perp \tilde{X}^i_{(3,1/2)} \perp \mathcal{O}(e_1 + e_2 - 2e_3)$$

with indecomposable summands of ranks 16, 9, and 1 respectively. The only direct summands of $(\tilde{J}^{(i)})_0$ of dimension ≤ 13 are $\tilde{X}^i_{(3,1/2)}$, $\mathcal{O}(e_1 + e_2 - 2e_3)$ and $\tilde{X}_{(3,1/2)} \perp \mathcal{O}(e_1 + e_2 - 2e_3)$, which have rank 9, 1, and 10, respectively. We thus have

$$\begin{aligned} \mathcal{E}_1 &= \tilde{X}^i_{(3,1/2)} && \text{if } \dim V_1 = 9 \\ &= \mathcal{O}(e_1 + e_2 - 2e_3) && \text{if } \dim V_1 = 1 \\ &= \tilde{X}^i_{(3,1/2)} \perp \mathcal{O}(e_1 + e_2 - 2e_3) && \text{if } \dim V_1 = 10. \end{aligned}$$

A similar consideration shows that

$$(\tilde{J}^{\{i+1\}})_0 = \mathcal{F}_1 \oplus \mathcal{F}_2 \qquad \text{with } \mathcal{F}_j \text{ a } GL(V_j)\text{-bundle}$$

with

$$\begin{aligned} \mathcal{F}_1 &= \overline{X}^i_{(2,1/3)} && \text{if } \dim V_1 = 9 \\ &= \mathcal{O}(e_1 + e_3 - 2e_2) && \text{if } \dim V_1 = 1 \\ &= \tilde{X}^i_{(2,1/3)} \perp \mathcal{O}(e_1 + e_3 - 2e_2) && \text{if } \dim V_1 = 10. \end{aligned}$$

Since $J^{(i)} \equiv J_0 \equiv J^{\{i+1\}} \mod(X, Y)$, we have $\mathcal{E}_1 \equiv V_1 \equiv \mathcal{F}_1 \mod(X, Y)$. Since $X_{(3,1/2)} \equiv X_2 \perp k(e_1 - e_2) \mod(X, Y)$ and $X_{(2,1/3)} \equiv X_2 \perp k(e_1 - e_3) \mod(X, Y)$, we have the following

(i) $k(e_1 + e_2 - 2e_3) = k(e_1 + e_3 - 2e_2)$ if $\dim V_1 = 1$

(ii) $X_3 \perp k(e_1 - e_2) = X_2 \perp k(e_1 - e_3)$ if $\dim V_1 = 9$

(iii) $X_3 \perp k(e_1 - e_2) \perp k(e_1 + e_2 - 2e_3) = X_2 \perp k(e_1 - e_3) \perp k(e_1 + e_3 - 2e_2)$ if $\dim V_1 = 10$.

Each of the above equalities clearly leads to a contradiction. This proves the proposition. ∎

THEOREM 7.8. *The Jordan algebras $J^{[i]}$, constructed in Section 6, give rise to $G = \mathrm{Aut}\, J$-bundles $P_{J^{[i]}}$ which are mutually non-isomorphic and specialize at any point of $\mathbb{A}^2_k$ to P_J. Further, they do not admit reduction of the structure group to any proper connected reductive subgroup H of G.*

Proof. We only need to show that $(J^{[i]})_0$ has no reduction of the structure group to any proper connected reductive subgroup H of G.

Suppose it admits such a reduction. Since H acts reducibly on the space V of trace zero elements of J, (7.5), $V = V_1 \oplus V_2$ with $\dim_k V_i \geq 1$ and each V_i is H-invariant. Hence $H \hookrightarrow (GL(V_1) \times GL(V_2)) \cap G$. This contradicts (7.7) and completes the proof. ∎

Remark 7.9. Suppose that the trace form Q restricted to the trace zero elements of J is anisotropic. This is the case, for example, if J is an anisotropic Jordan algebra over a real closed field or a number field. Then the Jordan algebra $J^{[i]}$ constructed in Section 6 has the further property that the trace form restricted to the 26 dimensional sub-space of trace zero elements is rigid. This can be proved by using a subtler variant of (2.6).

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